JOURNAL OF DIFFERENTIAL EQUATIONS **137**, 94–118 (1997)
ARTICLE NO. DE973260

The Problem of Center for Resonant Singular Points of Polynomial Vector Fields*

Henryk Żołądek†

Institute of Mathematics, University of Warsaw, 02-097 Warsaw, Banacha 2, Poland

Received July 11, 1996; revised October 24, 1996

We generalize the notion of "center" to the case of a $p : -q$ resonant singular point of a polynomial vector field in $\mathbb{C}^2$ and to some other situations (resonant node, saddle-node, non-elementary singular point of vector field and resonant fixed point of one-dimensional complex diffeomorphism). We show some similarities and differences with the classical case. In particular, the analogue of Bautin's theorem does not hold. Four small amplitude limit cycles can bifurcate from the center after quadratic perturbation of a quadratic vector field with resonant center.

1. INTRODUCTION

The problem of the center for polynomial vector fields with an elementary singular point

$$\dot{z} = (i + \lambda)\, z + \cdots, \qquad z = x + iy \in \mathbb{C}^2(\approx \mathbb{R}^2) \tag{1}$$

is the subject of much work. One constructs (step-by-step) the Lyapunov function $\Phi(z, \bar{z}) = z\bar{z} = \ldots$ satisfying the equation (for $\lambda = 0$) $\dot{\Phi} = v_3\, |z|^4 + v_5\, |z|^6 + \cdots$, where the coefficients v_i (called the Poincaré–Lyapunov focus quantities) are polynomials of the coefficients of the right-hand-side of (1). The theorem of Lyapunov and Poincaré [11, 12] says that when $\lambda = 0$ and all $v_i = 0$ then Φ is an analytic function forming a first integral of the system (1) and the point $z = 0$ is a center. This holds for germs of real analytic vector fields, where the condition for the center is given by the infinite series of equations $\lambda = v_1 = \cdots = 0$. If we are dealing with polynomial vector fields of degree n then the center conditions are reduced to a finite number of equations. For the quadratic case $n = 2$ they were given by

* Supported by Polish KBN Grant 2 P03A 022 08.

† Also at Institute of Math., Polish Academy of Sciences, skr. poczt. 137, 00-950 Warsaw, Śniadeckich 8, Poland. E-mail: zoladek@mimuw.edu.pl.

0022-0396/97 $25.00

Dulac [6] and Kapteyn [10]. For the cubic case $n = 3$ the complete center conditions are not known.

Connected with the above is the problem of small amplitude limit cycles: find the maximal number $M(n)$ of small limit cycles of the system (1) bifurcating from the origin when we start t vary the coefficients of the right-hand-side. For $n = 2$, Bautin [3] has shown that $M(2) = 3$. It is also known that $M(3) \geqslant 11$ (see [19]).

Let us note that the above problems are not necessarily real. After complexification the system (1) transforms to an analytic system in $\mathbb{C}^2$ with eigenvalues $+i$ and $-i$. This is $1 : -1$ resonant singular point and the numbers v_j become the coefficients before the resonant terms in its orbital normal form.

This was the way chosen by Dulac [6]. For the system

$$\dot{x} = \lambda x + A_1 x^2 + B_1 xy + Cy^2, \qquad \dot{y} = -y + Dx^2 + A_2 xy + B_2 y^2 \tag{2}$$

he calculated the focus numbers (for $\lambda = 1$) using the Lyapunov function $\Phi(x, y) = xy + \cdots$ satisfying the condition $\dot{\Phi} = g_1 (xy)^2 + g_2 (xy)^3 + \cdots$. He obtained

$$g_1 = -2(A_1 B_1 - A_2 B_2),$$
$$g_2 = -\tfrac{2}{3}[(2A_1 + A_2)(A_1 - 2A_2)\, A_2 C - (2B_2 + B_1)(B_2 - 2B_1)\, B_1 D],$$
$$g_3 = -\tfrac{5}{4}(A_2 B_1 - CD)[(2A_1 + A_2)\, A_2^2 C - (2B_2 + B_1)\, B_1^2 D].$$

These three numbers generate the whole ideal of coefficients g_j. From this the following list of $1 : -1$ *resonant centers* follows:

Q_3^{LV}: $B_1 = A_2 = 0$ (Lotka–Volterra systems with 3 invariant lines);

Q_3^H: $2A_1 + A_2 = 2B_2 + B_1 = 0$ (Hamiltonian systems);

Q_3^R: $A_1 B_1 - A_2 B_2 = A_2^2 C - B_1^3 D = 0$ (reversible systems);

Q_4: $A_1 - 2A_2 = B_2 - 2B_1 = A_2 B_1 - CD = 0$ (systems with invariant conic and cubic curves).

In other words the *center variety* (i.e. the set of systems with center) has 4 irreducible components.

One can extend Bautin's result to this case. The periodic trajectories in the real case form real ovals of complex solutions (Riemann surfaces) ϕ. They represent 1-dimensional cycles γ on these surfaces. They are limit cycles when they are isolated. The latter property can be described in terms of the monodromy (or holonomy) map Δ_γ, induced by such a cycle. It is defined as follows.

DEFINITION 1. Take a small holomorphic disc D transversal to the solution ϕ at a point $p \in \gamma$. If $x \in D$ then the loop γ has a unique lift to the solution passing through x with x as its beginning and $\Delta_\gamma(x)$ as the end.

A cycle γ is a limit cycle iff p is an isolated fixed point of any iteration Δ_γ^N. Of course, we must identify cycles which lie on the same solution and are homotopic to one another.

With the singular point $\dot{x} = x + \cdots$, $\dot{y} = -y + \cdots$ one can associate also another monodromy map. There are two (complex) analytic separatrices S and U passing through the origin, invariant stable and invariant unstable manifolds. Take a point $p \neq (0, 0)$ in one of them (say in S) and a disc D_S transversal to S. As above we define the holonomy map $\Delta_S \colon D_S \to D_S$ corresponding to a small loop γ in S starting at p and surrounding the origin. Analogously we define $\Delta_U \colon D_U \to D_U$.

The real Poincaré return map can be interpreted as follows. Take a small disc $D = \{x = y, |x| < \varepsilon\}$ and a monodromy map $\Delta \colon D \to D$ defined by means of a lift to the solution ϕ passing through $(x_0, x_0) \in D$ of the loop $\{x = x_0 e^{i\theta} \colon 0 \leqslant \theta \leqslant 2\pi\}$ in U. The center corresponds to the situation with $\Delta \equiv \mathrm{id}$ and we obtain a holomorphic 1-parameter family of (non-homologous to zero and small) cycles $\{\gamma_c\}$ (laying in different phase curves ϕ_c) meeting D at two points: (x, x) and $(-x + \cdots, -x + \cdots)$. The cycles are parametrized by a ring, the quotient $D^*/\mathbb{Z}_2$, $D^* = D \backslash \{0\}$. This picture can serve as the geometrical definition of a $1 : -1$ resonant center.

The focus case corresponds to the situation when 0 is an isolated fixed point of the monodromy map Δ. The order of the first non-zero term in the Taylor expansion of $\Delta - \mathrm{id}$ is the order of the focus.

The small limit cycles correspond to isolated periodic trajectories of Δ_S. In the $1 : -1$ resonance case we are interested in limit cycles which are small perturbations of the cycles $\gamma_s = \{(ze^{i\theta}, ze^{-i\theta}) \colon 0 \leqslant \theta \leqslant 2\pi\}$ so the above periodic trajectories became fixed points of Δ_S (outside S). After a local analytic change of coordinates we can assume that the separatrices form the coordinate axes $S = \{y = 0\}$, $U = \{x = 0\}$ and we can parametrize D_S by $xy = \mathrm{const} \cdot y$. Then from the results of Dulac [6] and from the radicality of the Bautin ideal, (i.e. the ideal generated by $g_1, g_2, \ldots$, see [17]), using the arguments of Bautin [3] we get that there can be at most 3 limit cycles bifurcating from the origin after perturbation. We provide a proof of this in Appendix.

(Note that there are always two cycles at the separatrices, which are not taken into account by us. There can also be arbitrarily many limit cycles corresponding to periodic trajectories of Δ_S with large period. That happens when the ratio of eigenvalues becomes a rational number with large denominator.)

The next natural generalization of the above theory is to consider the case of a polynomial vector field in $\mathbb{C}^2$ with $p : -q$ resonant elementary singular point

$$\dot{x} = px + \cdots, \qquad \dot{y} = -qy + \cdots$$

$p, q \in \mathbb{Z}$.

If $p > 0$, $q > 0$, $(p, q) = 1$, then we can repeat the above. The linear part has analytic first integral $H_0 = x^q y^p$ (with the integrating factor $M_0 = x^{q-1} y^{p-1}$) and we seek the conditions for the existence of an analytic first integral $\Phi = H_0 + \cdots$. We get the equation

$$\dot{\Phi} = g_1 H_0^2 + g_2 H_0^3 + \cdots$$

and the $p : -q$ *resonant focus number* $g_1, \ldots$ are polynomials of the coefficients of the vector field. One can easily check that

(i) deg $g_i = (p+q)\, i$ and

(ii) g_i are invariant with respect to the quasi-homogeneous action of $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ on the coefficients induced by: $x \to \mu^{-p} x$, $y \to \mu^q y$. (This fact corresponds to the invariance of v_i's with respect to rotations).

We can also interpret the g_i's as the corresponding coefficients in the (formal and not orbital) normal form of the resonant saddle

$$(px\partial_x - py\partial_y)(1 + h_1 H_0 + h_2 H_0^2 + \cdots)$$
$$+ (px\partial_x + qy\partial_y)(g_1 H_0 + g_2 H_0^2 + \cdots)$$

In the orbital normal form the coefficients h_i, which can be interpreted as the *resonant non-isochronicity numbers*, can be chosen equal to zero.

If all $g_i = 0$ then the center is defined by the family $\gamma_z = \{(ze^{pi\theta}, ze^{-iq\theta} + \cdots)\}$ of cycles. (Note that the center is *isochronous* iff $\int_{\gamma_x} dt \equiv \text{const.}$)

One can also consider the case of $1 : 0$ resonance, i.e. the saddle-node, and the case with $q < 0 < p$, i.e. the node. In the case of a node we are in the Poincaré domain (see [1]) and there are two possibilities:

(i) $p > 1$, $|q| > 1$, $(p, q) = 1$ where the system is analytically linearizable with meromorphic first integral;

(ii) $p = 1$ and there is only one resonant term in the normal form $\dot{x} = x$, $\dot{y} = |q|\; y + g_1 x^{|q|}$, with g_1 being a polynomial of the coefficients.

When there is a meromorphic first integral $\tilde{x}^q \tilde{y}^p$ then each local solution $\tilde{y} = C\tilde{x}^{|q|}$, $C \in \mathbb{C}^*$ forms an analytic image of a punctured disc with a generator γ_c of its fundamental group. This is a family of cycles defining the center.

One can also easily associate certain family of cycles in the case of a saddle-node center (see the next section).

Because we consider only elementary singular points with resonance, the above cases are all which have to be taken into consideration. Now we can give the definition (coming from Dulac [6]) of a resonant center or focus.

DEFINITION 2. A $p:-q$ resonant elementary singular point of an analytic vector field is a center iff there exists a local meromorphic first integral $H=H_0+\cdots$, $H_0=x^q y^p$. Equivalently: iff there is a holomorphic family $\{\gamma_c\}$ of small non-contractible loops on different phase curves ϕ_c of the vector field.

This singular point is a resonant focus of order k iff there is a formal power series $\Phi=x^q y^p+\cdots$ with the property $\dot{\Phi}=g_k H_0^{k+1}+\cdots$.

The following natural questions arise: determine the center conditions and describe the mechanism leading to the local integrability, study the neighbourhood of the center variety by means of the Dulac ideal, find the number of small limit cycles (perturbations of the cycles appearing in the geometrical definition of center), study the isochronicity of centers.

Another question is the generalization of the above to non-elementary singular points. We discuss this subject in Section 4.

2. THE RESULTS

We classify our with respect to the kind of resonance $p:-q$ and with respect to the degree n of the vector field

$$\begin{aligned}\dot{x}&=px+A_1x^2+B_1xy+Cy^2+\cdots\\ \dot{y}&=-qy+Dx^2+A_2xy+B_2y^2+\cdots\end{aligned} \tag{3}$$

The natural quasi-homogeneous gradation of the coefficients $A=(A_1, A_2)$, $B=(B_1, B_2)$, $C, D, \ldots$ is the following: $w(A)=p$, $w(B)=-q$, $w(C)=-p-2q$, $w(D)=2p+q$.

2.1. *The Node Case* $1<p<-q$

THEOREM 1. *In this case there is always a center and no limit cycles appear after perturbations.*

Proof. From the theory of normal forms (see [1]) it follows that any analytic vector field

$$\dot{x}=\lambda x+\cdots, \qquad \dot{y}=\mu y+\cdots$$

close enough to the vector field (3) is locally analytically linearizable $\dot{\tilde{x}} = \lambda \tilde{x}$, $\dot{\tilde{y}} = \mu \tilde{y}$ with the first integral $\tilde{x}^{-\mu} \tilde{y}^{\lambda}$ which is meromorphic for $\lambda = p$, $\mu = -q > 0$. If $\lambda/\mu = p/|q|$, then all cycles close to γ_c (from the defining family) are non-isolated. If $\lambda/\mu \neq p/|q|$, then there are no such cycles. ∎

Remark 2.1. With the change of the ratio λ/μ of eigenvalues there remain cycles at the separatrices. They become limit cycles for λ/μ not rational. However we do not take them into account by the above agreement.

2.2. *The Node Case with* $p = 1 < -q$

THEOREM 2. *In this case there is only one focus number and the cyclicity of the center is* 1.

Proof. We reduce the perturbed system to the local analytic normal form

$$\dot{\tilde{x}} = \tilde{x}, \qquad \dot{\tilde{y}} = (|q| + \varepsilon)\, \tilde{y} + g_1 \tilde{x}^{|q|}$$

and calculate the derivative of $H_0 = \tilde{y} \tilde{x}^{-|q|}$:

$$\dot{H}_0 = \varepsilon H_0 - g_1$$

From this the result follows. ∎

EXAMPLES. The values of the first focus number g_1 are calculated for small values of $|q|$ and n. We have

$g_1 = D_0$ for $|q| = 1$ and for the system $\dot{x} = x + \cdots$, $\dot{y} = y + D_0 x + \cdots$

$g_1 = D$ for $|q| = 2$

$g_1 = (2A_1 - A_2)\, D$ for $|q| = 3$

$g_1 = [2(A_2 - 2A_1)(A_2 - 3A_1) + (B_2 - 2B_1)\, D]\, D$ for $|q| = 4$, $n = 2$

$g_1 = D \cdot h(A, B, C, D)$ for $|q| > 4$, $n = 2$,

where h is a polynomial of degree $q - 2$ which is also quasi-homogeneous of degree $q - 2$.

The center variety consists of only one component ($D_0 = 0$ or $D = 0$) for $|q| = 1, 2$. It has two components ($D = 0$ and $h = 0$) for $|q| = 3, 4$, $n = 2$. It has at least two component for $|q| > 4$, $n = 2$.

We suppose that the polynomial h is irreducible for $n = 2$ and all $|q| > 4$.

Remark 2.2. There is the interesting question of finding the first integral. Can it be expressed in terms of elementary functions (i.e. elementary first integral) or can it be expressed by quadratures (i.e. Liouvillian first integral) or maybe we can express it by means of some special functions? In the case $p=|q|=1$, $n=2$ the change $(x, y) \to (x, u)$, $u=y/x$ gives the linear equation $dx/d\alpha = ((A_1+B_1\alpha+Cu^2)\,x+1)/(D+(A_2-A_1)\,\alpha+B_2 - B_1)\,u^2 - C\alpha^3)$ with the Darboux–Schwartz–Christoffel integral

$$x(u-u_1)^{a_1}(u-u_2)^{a_2}(u-u_3)^{a_3}+\frac{1}{C}\int^{\alpha}(s-u_1)^{a_1-1}(s-u_2)^{a_2-1}(s-u_3)^{a_3-1}\,ds$$

In other cases we cannot write down the formula for the first integral. The main reason is that the $p:|q|$ resonant node is obtained from a general (non-integrable) polynomial system by squeezing some rational curve to a point in a process reverse to the desingularization.

2.3. *The Saddle-Node Case* $p=1, q=0$

THEOREM 3. (a) *Each case of a center corresponds to the situation with algebraic curve* $K(x, y)=0$ (*smooth at the center*) *consisting of non-isolated critical points. The center variety consists of n components, any one of which is characterized by the degree k of the curve* $K=0$ *and can be parametrized by the coefficients of the polynomial K and by the coefficients of a vector field of degree* $n-k$ (*obtained by division of* (3) *by K and non-singular at the center*).

(b) *The cyclicity of any center from the k th component is* $k(2n-k)$, $k=1, 2, ..., n$.

Proof. The degenerate singular point $\dot{x}=x+\cdots$, $\dot{y}=\cdots$ has an analytic unstable manifold. It has also a center manifold but it can be non-analytic (see [1]). The system is formally equivalent to the following one

$$\dot{\tilde{x}}=\tilde{x}(1+\cdots)$$
$$\dot{\tilde{y}}=g_0+g_1\tilde{y}+g_2\tilde{y}^2+\cdots$$

where $g_0=g_1=0$ and $g_2, g_3, \ldots$ are the focus numbers and the curve $\tilde{x}=0$ is the formal center manifold. The center case corresponds to the situation with all $g_i=0$, the existence of formal curve $\tilde{x}=0$ "consisting" of critical points (or critical point is of infinite codimension). But the system is analytic and this formal curve must be analytic. So, the function $K(x, y)=\tilde{x}$ is a polynomial. After division we obtain a non-vanishing vector field with analytic first integral $H=y+\cdots$. The family of limit cycles defining the center are cycles in the punctured discs $\{H(x, y) = c, 0 < |\tilde{x}| < \varepsilon\}$ parametrized by c from a small disc. This gives the point (a).

(b) After perturbation there appear singular points with unstable separatrices containing limit cycles. The maximal number of such singular points is the cyclicity.

LEMMA. *Let the polynomials $K = K_k(x, y)$, $R = R_{n-k}(x, y)$, $S = S_{n-k}(x, y)$ of indicated degrees be such that $K(0) = 0$, $\nabla K(0) \neq 0$, $(R, S)(0) \neq (0, 0)$. The maximal number of solutions $x_i(\varepsilon)$ tending to 0 as $\varepsilon \to 0$ of the system of algebraic equations*

$$KR + \varepsilon P = KS + \varepsilon Q = 0$$

with $P = P_n(x, y; \varepsilon)$, $Q = Q_n(x, y; \varepsilon)$ polynomials of degree n with analytic coefficients is equal to

$$k(2n - k)$$

Proof. We can assume that $K = x + \cdots$, $R = 1 + \cdots$, $Q(0) = 0$. From the first equation we have $K = -\varepsilon P + O(\varepsilon^2)$. Thus $P_n S_{n-k} - Q_n \approx 0$ and because we are near the curve $K = 0$ the needed points are approximately the solutions of the system of algebraic equations $K_k = P_n S_{n-k} - Q_n = 0$. Now we apply Bezout's Theorem. ∎

EXAMPLE. For $n = 2$ the first focus numbers are

$$g_2 = B_2, \qquad g_3 = -A_2 C, \qquad g_4 = C^2 D \pmod{g_3}$$

and there are two cases of center

$$B_2 = C = 0 \text{ with cyclicity 3 and}$$

$$A_2 = B_2 = D = 0 \text{ with cyclicity 4.}$$

Indeed, we can perform calculations. Let the center manifold be given by the equation

$$x = ay^2 + by^3 + \cdots$$

One finds that $a = -C$, $b = C(2B_2 - B_1)$. Then the vector field restricted to the center manifold is

$$\dot{y} = B_2 y^2 - A_2 C y^3 + [C^2 D + A_2(2B_2 - B_1)\, C]\, y^4 + \cdots$$

From this the above property follows. ∎

Remark 2.3. For $n=2$ all the cases of center are integrable (by means of Darboux integrals, see [15, 17]). For higher degrees it is not the case.

2.4. *The Resonant Saddle* $1 \leqslant p < q$

Assume that $(p, q)=1$. Here we have only results for the quadratic case. They are not complete.

THEOREM 4. *Let* $n=2$. *The maximum codimension of a component of the center variety is* 5 *and the maximal cyclicity is* $\geqslant 4$.

We will prove this theorem in the next section. Here we stress the difference between this case and the $1:-1$ resonant case. In Dulac's case the maximal cyclicity of the quadratic center is 3 and each component of the center variety contains parameters (moduli) which are independent on affine changes of variables and time. The same holds also in all known cases of cubic centers (see [14, 18]). In the $p:-q$ resonant cases some systems with center are isolated in the space of orbits of the group of affine equivalences. An important problem is to find some general explanation of this phenomenon.

The first nontrivial case is the case of $1:-2$ resonance. One can try to calculate the focus quantities. (We need just four of them). The first one is the following

$$g_1 = (A_1 B_2 + A_2 B_2 - 2A_1 B_1)\, A_2 - \tfrac{1}{2}(B_2 + B_2)(2B_1 - B_2)\, D \\ - \tfrac{1}{10}(12A_1 - A_2)\, CD$$

The significant progress in computing the other focus numbers and finding the cases of center was made by Fronville [8]. The first three quantities were also calculated by Romanovski [13].

The author has found several cases of centers for $p:-q$ resonances; they are given in the next theorem. The expressions for the first integrals below are given in some special affine coordinates and are of generic type. $F_1(z) = F(a, b; c; z) = 1 + (ab/c1!)\, z + ((a(a+1)\, b(b+1))/(c(c+1)\, 2!))\, z^2 + \cdots$ is the hypergeometric function (see [2]), $F_2(z) = F(a-c+1, b-c+1; 2-c; z)$.

THEOREM 5. *The following conditions and functions define the sufficient center conditions and the corresponding first integrals for quadratic vector fields with* $p:-q$ *resonant saddle*, $0<p<q$:

(1) $A_2 = D = 0$; $x^{p/q}(([x + cy/(ab)]\, F_1(y) - y(1-y)\, F_1'(y))/$
$([p/q + z + (c/(ab) - p/q)\, y]\, F_2(y) - y(1-y)\, F_2'(y)))$,
$c = 1 + p/q$

(2) $B_1 = C = 0,\ p > 1;\ x^{q/p}(([y + cx/(ab)]\, F_1(x) - x(1-x)\, F_1'(x))/$
$([q/p + y + (c/(ab) - q/p)\, y]\, F_2(x) - x(1-x)\, F_2'(x))),$
$c = 1 + q/p$

(3_k) $p = 1,\ k = 1, ..., [(q+1)/2],\ B_2 = C = 0,\ [A_2 + (k-1)\, A_1]$
$[A_2 + (q-k)\, A_1] + ((q+1-2k)^2/k(q+1-k))\, DB_2 = 0;$
as above with $a = k,\ c = q+1$

(4) $p = 1,\ B_1 = B_2 = C = 0;$
$(1+x)^{-a}[1 + \binom{a}{1}\, x + \cdots \binom{a}{q+1}\, x^{q+1} + x^q y]$

(5) $(p+q)\, A_1 + pA_2 = qB_1 + 2pB_2 = C = 0;\ x^q(y + x^2 + axy + y^2)^p$

(6) $2qA_1 + pA_2 = qB_1 + (p+q)\, B_2 = D = 0;\ (x + x^2 + axy + y^2)^q\, y^p$

(7) $qA_1B_1 + (p-q)\, A_1B_2 - pA_2B_2 = C = D = 0;\ x^q y^p(1 + x + y)^a$

(8) $p = 1,\ q = 2,\ A_2 = B_1 + B_2 = C = 0;$
$(1 + x + xy - (a-2)\, x^2/6)^a/(1 + ax + axy + (2a-1)\, x^2/6)$

(9) $p = 1,\ q = 2,\ A_2 = 2B_1 - B_2 = C = 0;$
$((1 + y + (a+1)\, x)^a\, (1 + y + (a-2)\, x)^{1-a})/(1 + y + (4a-2)$
$x + 9a(a-1)\, x^2/2)$

(10) $A_1 = p(4p + 3q),\ A_2 = 2p(p + 2q),\ B_1 = 2q(2p + q),$
$B_2 = q(3p + 4q),\ C = q(p + 2q),\ D = p(2p + q);\ ([(x + y)^2$
$+ 2p(p+q)\, x]^q$
$[(x + y)^2 + 2q(p+q)\, y]^p)/(x + y + pq)^{2p+2q}$

(11) $A_1 = A_2 = C = 2B_1 - B_2 = 0;$
$(x^{2q}[x^2 + (2p+q)\, y]^{2p})/(x^2 + 2py + 1)^{2p+q}$

(12) $A_1 - 2A_2 = B_1 = B_2 = D = 0;$
$([y^2 + (p + 2q)\, x]^{2q}\, y^{2p})/(y^2 + 2qx + 1)^{p+2q}$

(13) $A_1 = pq,\ A_2 = -2q(2p+q),\ B_1 = 2p(p+2q),\ B_2 = -pq,$
$C = -q(p + 2q),\ D = p(2p+q);\ [q + (x + y - 1)$
$(-(2p+q)\, x + qy + q)]^q[p + (x + y - 1)$
$(px - (p+2q)\, y + p)]^p$

(14) $A_1 = -p,\ A_2 = q - p,\ B_1 = D = 0,\ B_2 = p + 3q,\ C = -p - 2q;$
$([y + (x + y + 1) - (x + y + 1)^2]^q\, y^p)/$
$[(p+q)\, y + q(x + y + 1)]^{p+q}$

(15) $A_1 = -3p - q,\ B_1 = q - p,\ A_2 = C = 0,\ B_2 = q,\ D = q + 2p;$
$(x^q[x + (x + y + 1) - (x + y + 1)^2]^p)/$
$[(p+q)\, x + p(x + y + 1)]^{p+q}$

Remark 2.4. The present form of the conditions (1) and (2) is due to calculations performed by A. Fronville (Paris VI). Initially the author knew some subcases of these cases. The vanishing of the first two focus numbers (calculated by Fronville) forced the author to look for the first integral.

In the case of $1:-2$ resonance, A. Fronville has found several other cases, for which the first five quantities are zero. It is very probable that the list of the center cases for this resonance is complete. However the formulas for the corresponding first integrals are not known. These additional cases are the following:

$$A_1+2A_2=B_1=B_2=D=0$$

$$A_1+A_2=B_1=D=0$$

$$A_1=3, \qquad A_2=1, \qquad B_1=3, \qquad B_2=2, \qquad C=0, \qquad D=-1$$

$$A_1=11, \qquad A_2=2, \qquad B_1=30, \qquad B_2=12, \qquad C=0, \qquad D=-1$$

$$A_1=-19, \qquad A_2=-4, \qquad B_1=B_2=-14, \qquad C=5, \qquad D=10$$

$$A_1=-2, \qquad A_2=1, \qquad B_1=b, \qquad B_2=2b+2, \qquad C=-2b, \qquad D=1$$

$$A_1=-29, \qquad A_2=28, \qquad B_1=-2, \qquad B_2=34, \qquad C=-5, \qquad D=10$$

$$A_1=7, \qquad A_2=4, \qquad B_1=-2, \qquad B_2=-8, \qquad C=-5, \qquad D=4$$

$$A_1=13(1\pm\sqrt{6}), \qquad A_2=10, \qquad B_1=52\mp 22\sqrt{6},$$
$$B_2=2\mp 18\sqrt{6}, \qquad C=2(1\pm\sqrt{6}), \qquad D=5$$

Proof. We do not repeat the rather cumbersome calculations and strive to give the reader some ideas why each case gives the resonant center.

Because the cases (1) and (2) are analogous we focus our attention on the second one. After some reductions we obtain the Ricatti equation

$$\frac{dy}{dx}=\frac{Kx}{1-x}-\frac{Lx+\lambda}{x(1-x)}\,y+\frac{y^2}{x(1-x)},$$

$\lambda=q/p$. Using the change of variable $y=-x(1-x)\,z'/z$, the Ricatti equation becomes associated with the second order linear equation

$$x(1-x)^2\,z''+[(L-2)\,x-(\lambda+1)](1-x)\,z'+kxz=0$$

The substitution $z=(1-x)^{\alpha}\,w$ gives the hypergeometric equation (see [2])

$$x(1-x)\,w''+[c-(a+b+1)\,x]\,w'-abw=0,$$

where $\alpha^2-(L+\lambda)\,\alpha+K=0$, $c=\lambda+1$, $ab=\alpha(\lambda+1)$, $a+b=1+2\alpha-L$. The hypergeometric equation has two independent solutions $F_1(x)=F(a,b;c;x)$ and $x^{1-c}F_2(x)=x^{1-c}F(a-c+1,b-c+1;2-c;x)$. Now the general solution of the Ricatti equation is $y = -x(1-x)((C_1 z_1'(x) + C_2 z_2'(x))/(C_1 z_1(x) + C_2 z_2(x)))$, where $z_{1,2}$ are two independent solutions of the linear equation. Expressing the ratio C_2/C_1 in terms of x, y we obtain the formula for the first integral, which is analytic near the origin because the hypergeometric functions are analytic near $x=0$.

The point (3_k). If c is a positive integer then the function F_2 from the previous case may contain the term $\log x$. This term disappears iff $a = 1, 2, ..., c-1$ or $b = 1, 2, ..., c-1$. Assume that $a = k$ is an integer. Notice now that the center conditions are invariant with respect to the change $k \to q+1-k$. The function F_2 is rational and there is another expression for the first integral of the Darboux type (which we do not present here).

In the point (4) we have a Darboux first integral which has the line $x=0$ as the curve of non-isolated critical points: $H(x, y) - 1 = x^2(y + \cdots)$.

The points (5), (6), (7) are rather obvious.

In the points (8), (9) again there is the line $(x=0)$ of non-isolated critical points.

The cases (10), (11), (12) arise from 1-parameter family of Darboux integrals $x^{2-2\alpha}(x^2+y)^{\alpha}(x^2+ay+b)^{-1}$ for a quadratic vector field; it is the quadratic case $C_{1,2,2}$ from the Sokulski's Thesis [15].

In case (13), the line $x+y=1$ is critical.

In cases (14) and (15) the line $x+y+1=0$ is critical. ∎

Remark 2.4. We see that sometimes the first integral is expressed by means of hypergeometric functions and the equation for the phase portrait becomes a Ricatti equation with non-solvable monodromy group. This integral cannot be expressed by quadratures. (Also among non-real $1:-1$ resonant polynomial centers the situation with Ricatti equation and non-quadrature first integral occurs quite naturally). In all known cases of real polynomial centers the system is either reversible or has Liouvillian first integral. (The author conjectures that this holds always, see [18]). However the above Ricatti systems do not seem to be reversible. This would mean another difference with the real center case.

Notice also that in each of the cases written above at least one of the separatrices of the resonant saddle is an algebraic curve. Maybe this is a rule for $p:-q$ resonance with $q>1$. In the case of $1:-1$ resonance both separatrices can be non-algebraic.

3. PROOF OF THEOREM 4

Let us consider case (12) of the $p:-q$ resonant center. It is of codimension 5 in the space of all quadratic systems: we have one condition for the ratio of eigenvalues and the four conditions $A_1 - 2A_2 = B_1 = B_2 = D = 0$. We can choose a first integral in the form

$$H = \frac{[y^2 + (p+2q)\,x]^{2q}\, y^{2p}}{(y^2 + 2qx + 1)^{p+2q}}$$

and the general perturbation of the form

$$\begin{aligned} \dot{x} &= px + 2pqx^2 + 2py^2 + \varepsilon_1 x + \varepsilon_2 x^2 + \varepsilon_3 xy \\ \dot{y} &= -qy + pqxy - \varepsilon_4 x^2 - \varepsilon_5 y^2 \end{aligned}$$

with small parameters ε_i. We consider the linear Poincaré–Pontriagin–Melnikov integral

$$I(h) = \int_{\gamma_h} \frac{\omega}{y[y^2 + (p+2q)\,x][y^2 + 2qx + 1]} = \sum \varepsilon_i I_i(h),$$

$\omega = (\varepsilon_1 x + \varepsilon_2 x^2 + \varepsilon_3 xy)\,dy + (\varepsilon_4 x^2 + \varepsilon_5 y^2)\,dx$, where the integral runs along the family of cycles

$$\gamma_h = \{x \sim h^{1/(2(p+q))} e^{ip\theta},\ y \sim h^{1/(2(p+q))} e^{-iq\theta} : 0 \leqslant \theta \leqslant 2\pi\},$$

$h \to 0$ on the curves $H(x, y) = h$ defining the $p : -q$ resonant center. (We have $\Delta H = \varepsilon h I(h) + O(|\varepsilon|^2)$ for the increment of the function H along trajectory of the perturbed system.) Theorem 4 follows from the following (as in [19]).

PROPOSITION. *The functions $I_1(h), ..., I_5(h)$ are independent.*

Proof. We apply the following change of variables

$$h = t^p, \qquad u = \frac{t^{1/2}}{y}, \qquad v = \left[t \left(1 + (p+2q)\,\frac{x}{y^2} \right) \right]^{1/(p+2q)}$$

Then the equation $H(x, y) = h$ can be rewritten as

$$F(u, v) = (p+2q)\,u^2 + 2qv^{p+2q} - (p+2q)\,v^{2q} = -pt$$

(Note that $h = t^p = [1 + (p+2q)\,x/y^2]^{2q}/[1 + 2qx/y^2 + 1/y^2]^{p+2q} = (v^{2q})^{p+2q} t^{-2q}/[2q/(p+2q)\cdot((v^{p+2q})/t) + (p/(p+2q)) + (u^2/t)]^{p+2q})$. The family of cycles γ_h can be represented as a family of cycle $\delta(t)$ in the following way. The curve $F(u, v) = -pt$ is the Riemann surface of the function

$$u = \sqrt{R(v)}$$

As $h \to 0$ (and $t \to 0$) $2q$ roots $v_1, ..., v_{2q}$ roots of the polynomial $R(v)$ are of order $\sim t^{1/(2q)}$ and the other roots $w_0, ..., w_{p-1}$ tend to some nonzero constants. The cycle γ_h becomes

$$\delta(t) = \{u \sim t^{q/(2p+2q)} e^{iq\theta}, v \sim t^{1/(2p+2q)} e^{-ip\theta}\}$$

Because $t^{1/(2p+2q)} \gg t^{1/(2q)}$ we see that $\delta(t)$ forms a (unique) lift to the Riemann surface of the loop surrounding $v_1, ..., v_{2q}$ (see Fig. 1).

The function $(-F(u, v))/p$ has $p+2$ critical values: 0, ∞, $t_j = e^{4\pi iqj/p}$, $j = 0, ..., p-1$. As t tends to 0, $2q-1$ cycles $\tau_j(t)$ (corresponding to loops surrounding the points v_j and v_{j+1}, $j = 1, ..., 2q-1$) vanish. As t tends to t_j, two points, w_j and one of v_i's approach each other and the corresponding cycle $\sigma_j(t)$ vanishes. As t tends to infinity all the points v_i and w_j tend to infinity.

The monodromy corresponding to a loop around $t = 0$ acts by cyclic permutation of the cycles $\tau_j(t)$; (here $\tau_{2q}(t) = -\Sigma\, \tau_j(t)$). The cycle $\delta(t)$ becomes invariant. The monodromy corresponding to loops around t_j keeps the cycles $\sigma_j(t)$ invariant and $\delta(t) \to \delta(t) + \sigma_j(t)$. The monodromy corresponding to the loop around $t = \infty$ permutes cyclically the cycles associated with the points $\sim t^{1/(p+2q)} e^{2\pi ik/(p+2q)}$ and $\sim t^{1/(p+2q)} e^{2\pi i(k+1)/(p+2q)}$, $k = 0$, ..., $p+2q-1$. From the above (as in [19]) we get the following.

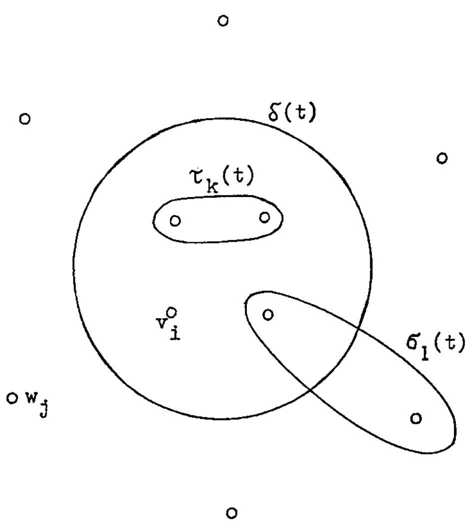

FIGURE 1.

LEMMA 3.1. *Let $\eta(u, v)$ be a rational 1-form. Then the function*

$$I_\eta(t) = \int_{\delta(t)} \eta$$

is meromorphic at $t = 0$. It is of the form $f(t) + \tilde{I}_\eta(t) \log(t - t_j)$ near t_j, where $f(t)$ and $\tilde{I}_\eta(t) = (1/2\pi i) \int_{\sigma_j(t)} \eta$ are meromorphic functions (near t_j). Near ∞ we have $I_\eta(t) = g(t^{-1/(p+2q)})$, where g is some meromorphic function.

Now we represent the integrals I_j in the new variables. Let

$$J_k(t) = \int_{\delta(t)} v^k u \, dv, \qquad J'_k(t) = \int_{\delta(t)} v^k \, dv/u$$

LEMMA 3.2. *We have*

$$\begin{aligned}
(p+2q)\, I_1 &= -2\pi i q \\
(p+2q)\, I_2 &= -2\pi i \left[\frac{p}{2q} - (p+2q) \sum_0^{p-1} (w_j^p - 1)^2 \, w_j^{-p} \right] \\
(p+2q)\, I_3 &= -2qt^{1/2} J_{-2q-1} + (p+4q)\, t^{3/2} J_{-p-4q-1} \\
(p+2q)^3\, I_4 &= +(3p+2q)\, t^{-1/2}\, J'_{2p+2q-1} - (4p-4q)\, t^{1/2} J'_{p-1} \\
&\quad - (p+14q)\, t^{3/2} J'_{-2q-1} - (2p+8q)\, t^{5/2} J'_{-p-4q-1} \\
(p+2q)\, I_5 &= -pt^{1/2} J_{-2q-1} - (2p+8q)\, t^{3/2} J_{-p-4q-1}
\end{aligned}$$

Proof. We make the substitution

$$x = \frac{v^{p+2q} - t}{(p+2q)\, u^2}, \qquad y = \frac{t^{1/2}}{u}, \qquad y^2 + (p+2q)\, x = \frac{v^{p+2q}}{u^2}$$

$$y^2 + 2qx + 1 = \frac{v^{2q}}{u^2}, \qquad dx = v^{p+2q-1} \frac{dv}{u^2} - \frac{2(v^{p+2q} - t)}{(p+2q)\, u^2}\, du, \qquad dy = -\frac{t^{1/2}}{u^2}\, du$$

and $u\, du = qv^{2q-1}(1 - v^p)\, dv$ to the formulas for I_j. We obtain $I = t^{-1/2} \int (u^5/u^{p/4q})\, \omega$.

So,

$$I_1 = \frac{-1}{p+2q} \int \frac{v^{p+2q} - t}{v^{p+4q}}\, u\, du = \frac{q}{p+2q} \int \frac{(v^{p+2q} - t)(1 - v^p)}{v^{p+2q+1}}\, dv$$

which can be calculated using the residue theorem.

Next

$$I_2 = -(p+2q)^{-2} \int \frac{(v^{p+2q}-t)^2\,du}{v^{p+4q}u}$$

where we represent du/u as $u\,du/u^2 = qv^{2q-1}(1-v^p)\,dv/R(v)$ and next calculate the residues at $v = w_i$ and at $v = \infty$.

We have

$$I_3 = \frac{-t^{1/2}}{p+2q} \int \frac{v^{p+2q}-1}{v^{p+4q}}\,du$$

and we integrate it by parts. In the same way we treat I_5.

Finally

$$I_4 = \frac{-t^{-1/2}}{(p+2q)^2} \int \frac{(v^{p+2q}-t)^2\,u}{v^{p+4q}} \left(\frac{v^{p+2q-1}\,dv}{u^2} - \frac{2(v^{p+2q}-t)}{(p+2q)\,u^3}\,du \right)$$

and using the integration by parts we get the conclusion of the lemma. ∎

We pass to the proof of the independence of I_j's. From Lemma 3.1 we get that the functions I_3, I_4, I_5 contain terms $\tilde{I}_j \log(t-1)$, $j = 3, 4, 5$, where $\tilde{I}_j$ are corresponding integrals along σ_0. The functions I_1 and I_2 do not have such terms. Therefore we strive to show that

(i) $\tilde{I}_3, \tilde{I}_4, \tilde{I}_5$ are independent and

(ii) I_1 and I_2 are independent.

Note that $\tilde{J}'_k \to A = \lim \int_{\sigma_1(t)} dv/u \neq 0$ as $t \to t_0 = 1$; (because the corresponding critical point of the function F is of Morse type). So $(p+2q)^3\, I_4 \to -(4p+16q)\,A \neq 0$. But $J_k \sim \int u\,dv \sim (t-1)$ and thus $\tilde{I}_4$ is independent on $\tilde{I}_3$ and $\tilde{I}_5$.

As $t \to \infty$ we have $u \sim t^{1/2}$, $v \sim t^{1/(p/2q)}$. Therefore $\tilde{J}_k \sim \text{const}\ t^{(k+1)/(p+2q)+1/2}$, where const depends on the cycle and is nonzero for a cycle surrounding just two roots of $R(v)$. (Due to the action of the monodromy group we can assume that they are just adjacent roots). Hence $\tilde{J}_{-2q-1} \sim t^{(p-2q)/(2p+4q)}$, $\tilde{J}_{-p-4q-1} \sim t^{(-p-6q)/(2p+4q)}$ are independent. This implies the independence of $\tilde{I}_3$ and $\tilde{I}_5$.

To show the property (ii) it is enough to notice that the function I_2 is a non-constant algebraic function of t. But as $t \to \infty$ we have

$$I_2 \sim \sum w_j^p \sim t^{p/(p+2q)} \left(\sum_0^{p-1} e^{(2\pi i p k_j)/(p+2q)} \right)$$

where the numbers $k_j \in \{0, ..., p+2q-1\}$ can be chosen arbitrarily due to the action of the group $\pi_1(\mathbb{C}\setminus\{0, t_0, ..., t_{p-1}\})$ on the roots of $R(v)$. Thus $I_2 \sim \text{const } t^{p/(p+2q)}$, const $\neq 0$.

This completes the proof of Proposition and of Theorem 4. ▮

4. NON-ELEMENTARY SINGULAR POINTS

4.1.

The geometrical definition of the center is useful here: the center is a 1-parameter family of small cycles $\{\gamma_c\}$ on solutions ϕ_c which are not contractible and not homotopic among themselves. We shall make this statement precise later.

The definition of small limit cycles appearing after perturbation of the center is more delicate. There may appear new singular points with new separatrices. Some of the new-born limit cycles may lay on these separatrices, other may lay outside of them. One possible definition of cyclicity would be restricted only to deformations which do not produce new critical points. The general definition should admit bifurcations of singular points and counts up all limit cycles close to the initial non-isolated cycles. The author prefers the second definition.

Let us recall that a singular point is non-elementary if and only if both of its eigenvalues are zero. Of course, in this situation Seidenberg's version of the resolution of singularity theorem must be applied (see [14]). We assume that the reader knows its formulation.

4.2.

Consider first the case when the resolution is obtained in one blowing-up. Then the singular point O is replaced by a projective line $\mathbb{C}P^1$ (the exceptional divisor) with the self-intersection index -1. We have two possibilities:

(i) the di-critical case, when the exceptional divisor is not invariant with respect to the blown-up field of directions;

(ii) the non-di-critical case, when the $\mathbb{C}P^1$ is invariant with several singular points $w_1, ..., w_p$.

In the case (i) almost all trajectories near O lie on invariant locally analytic invariant curves ϕ_c passing through O. (Almost all because there can be some singular and tangency points on the divisor $\mathbb{C}P^1$). We have a family of cycles γ_c in $\phi_c\setminus\{O\}$. The number of limit cycles appearing after perturbation is equal to the number of appropriate separatrices of singular points appearing after perturbation and can be effectively calculated.

If there is some center in the case (ii) then the blown-up family $\tilde{\gamma}_c$ of cycles (derived from the family γ_c) will accumulate along some loop $\gamma \subset \mathbb{C}P^1 \setminus \{w_1, ..., w_p\}$. When γ surrounds only one point w_i, then the center is the same as in the case of elementary resonant singularity. Otherwise the problem of the center becomes non-algebraic (see [10]). (However when we fix the lowest order part of the vector field then we have analogues of the focus numbers depending algebraically on the coefficients of the higher order terms of the vector field). Note that there can be several centers associated with one such singular point.

The problem of cyclicity of non-elementary centers (in the case (ii)) is not even touched in the real case. So, here lies an open field for investigation.

4.3.

When there are more blowing-ups in the desingularization process, the above construction still works. The singular point is replaced by a collection of exceptional divisors $\{E_j\}$ with one-point transversal intersections on some complex surface. Some of E_j's are not invariant (the resulting field of directions is in the di-critical case) and we have a center (or centers) associated with a 1-parameter family of analytic invariant curves passing through O.

Let $E' = \bigcup_{j \in K} E_j$ denote the union of invariant divisors and let $E^* = E' \setminus \bigcup_{j \in J} \{\text{critical points on } E_j \text{ different from } E_j \cap E_i\}$. If there is some 1-parameter family of cycles defining a center not associated with a di-critical divisor then these cycles accumulate along some loop γ in E^*.

On the other hand with any loop $\gamma \subset E^*$ one can adjoin a monodromy map Δ_γ. However the domain of its definition may be quite thin, like a sector in a disc.

Definition. (a) We say that a loop $\gamma \subset E^*$ is a center for the germ of vector field iff the associated monodromy map Δ_γ is the identity. In this case there is a family $\{\gamma_c\}$ of cycles at different phase curves ϕ_c of the vector field accumulating at γ.

Otherwise we say that γ represents a focus, whose order is defined as the order of its first non-zero term in the Dulac expansion of the map $\Delta_\gamma - \mathrm{id}$.

(b) Any di-critical divisor E_j is a center. In this case there is a family $\{\gamma_c\}$ of cycles at different phase curves $\phi_c = cl(\phi_c) \setminus E_j$ surrounding the point $cl(\phi_c) \cap E_j$ and of arbitrary small amplitude. (cl denotes the topological closure).

Medvedeva has developed some methods to study the real monodromy map Δ_γ, (see [4] for example). Probably one should try to adapt them to this situation.

4.4.

Consider the nilpotent singularity

$$\dot{x} = y + \cdots, \qquad \dot{y} = ax^2 + bxy + cy^2 + \cdots \tag{4}$$

When $a \neq 0$ then this is the Bogdanov–Takens singularity investigated (from the analytic point of view) by Cerveau and Moussu [5] and by Elizarov, Iliashenko, Scherbakov and Voronin [7]. Its desingularization gives three exceptional divisors E_1, E_2, E_3 as at Fig. 2. The point p_1 is a $1 : -2$ resonant saddle and the point p_2 is a $1 : -3$ resonant saddle. These two points are linearizable. It follows from the fact that the monodromy map associated with loops in E_1 and E_2 are identities, (the loops are contractible). So, there exist two families of cycles near p_1 and p_2, but they cannot give a center for the same contractibility reasons.

The point p_3 is a $1 : -6$ resonant saddle. If it is analytically linearizable then the family of cycles associated with it defines a center for the non-resolved initial singular point.

The monodromy group associated with $E_3 \setminus \{p_1, p_2, p_3\}$ is generated by two maps: $f(z) = -z + \cdots$ corresponding to a loop around p_1 and $g(z) = e^{2\pi i/3} z + \cdots$ (corresponding to a loop around p_2). We have

$$f^2 = g^3 = id$$

(Here $f^n = f \circ \cdots \circ f$). The loop around p_3 generates the map $f \circ g$. The point p_3 is linearizable iff $(f \circ g)^6 = \mathrm{id}$. But Theorem 4.2 from [7] asserts that the latter happens iff the monodromy group is solvable. Thus we have the following.

THEOREM 6. *If the monodromy group associated with E_3 is solvable then the system* (4) *has center.*

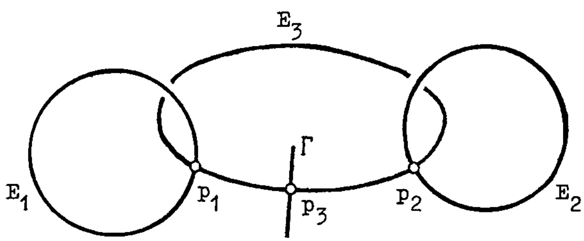

FIGURE 2

5. RESONANT FIXED POINTS FOR ALGEBRAIC DIFFEOMORPHISMS

Consider a germ of conformal diffeomorphisms of $(\mathbb{C}, 0)$

$$w = f(z) = \lambda z + \cdots$$

Assume that $\lambda = \lambda_0 = e^{2\pi i p/q}$, $p, q \in \mathbb{Z}$, $(p, q) = 1$ $q > 0$. Then the normal form, with respect to formal conjugation, of this map is the following

$$z \to \lambda z(1 + g_1 z^q + g_2 z^{2q} + \cdots)$$

where g_i can be interpreted as the "focus" numbers associated with the resonant fixed point of the diffeomorphism f. If $\lambda = \lambda_0$ and all $g_i = 0$ then f is analytically conjugate to a rotation or $f^q(z) - z \equiv 0$. We say that the equations $g_i = 0$ define the "center" case.

One can also introduce the analogue of the Lyapunov function $\Phi(z)$. In the "center" case it should be constant on the trajectories of the dynamical system defined by the iterations of f. So, $\Phi(z) = \Phi(f(z)) = \Phi(f^2(z)) = \cdots$. Moreover one should start calculation of Φ with z^q. Thus we have

$$\Phi(z) = z^q + \cdots$$

and

$$\Phi(f(z)) - \Phi(z) = g_0 z^q + q g_1 z^{2q} + \cdots$$

where $g_0 = \lambda^q - 1$. The equation defining the map $w = f(z)$ and its iterations in the "center" case would be

$$\Phi(z) = \Phi(w) = 0$$

Assume now that the situation is algebraic. We must make this notion precise. The case when $f(z)$ is given by a polynomial is not very interesting. Note that $f^q(z) \equiv z$ for polynomial f iff f is a linear map.

Very interesting series of examples give us algebraic relations

$$F(z, w) = 0 \tag{5}$$

where F is a polynomial of degree n and (5) defines $w = f(z)$ implicitly as an algebraic function. In general the equation (5) defines near $z = w = 0$ an analytic curve with (maybe) several branches. So, we assume that one of these branches S_0 is smooth at $(0, 0)$ and has the form $w = \lambda z + \cdots$. Just this branch defines the map f. In fact the map f is defined by the irreducible component $F_0(z, w) = 0$ of the reaction (5) containing the branch S_0.

EXAMPLE. The phase flow maps, $w = g^t(z)$, associated with the vector field $\dot{z} = z^{p+1}$ are of the form

$$(1 - ptz^p)\, w^p - z^p = 0$$

Also the maps μg^t, which form the solvable group $G_s(p)$ (see [7]), are given by algebraic relations.

The conditions for the "center" can be formulated algebraically. If two relations are algebraic, $R_1 = \{F(z, w) = 0\}$ and $R_2 = \{G(z, w) = 0\}$, then their composition is also algebraic: $R_2 \circ R_1 = \{(z, t) : \exists_w\, F(z, w) = 0, G(w, t) = 0\}$. Thus we have the following property.

PROPOSITION. *The relation* (5) *defines a resonant "center" of order q iff the q th iteration of the relation $F_0(z, w) = 0$ is given by an algebraic curve containing the line $z = w$ as its component.*

So, the "center" problem is solved. One can write down explicit equations for the coefficients of the function F of degree n. For $\lambda_0 = e^{2\pi i p/q}$ and n fixed these equations are algebraic and define an algebraic "center" variety.

There remains the problem of small "limit cycles" bifurcating from the "center". In the present situation the role of small cycles is played by isolated periodic orbits of f of period exactly q. Using the Lyapunov function, or other more tricky methods, we can calculate the "focus numbers" and "cyclicities" of the corresponding "centers" of "foci" in the class of maps given by (5) with fixed n and λ_0.

The author has not proceeded very far in these calculations. He hopes to lure other people to join this problem, which seems to be simpler than the standard centre-focus problem. Below we present only some simple examples.

EXAMPLES. 1. Let $q = 1$. Then the "center" conditions mean that $F(z, w)$ is divisible by $w - z$. Let

$$F_\varepsilon(z, w) = \lambda w - z + \sum_{i+j=2}^{n} A_{ij} z^i w^j \tag{6}$$

where $A_{ij} = A_{ij}(\varepsilon)$, $\lambda = \lambda(\varepsilon)$, $\lambda(0) = 1$. Then the focus numbers are

$$g_0 = \lambda - 1, \qquad g_k = \sum_{i+k=k+1} A_{ij}$$

and the maximal "cyclicity" is equal to $n - 1$. This holds when one considers only perturbations of the form (6) with $z = 0$ fixed point for f.

If

$$F_0(z, w) = (w - z)\left(\sum_{i+j-r} B_{ij} z^i w^j\right) + \sum_{i+j=r+2}^{n} A_{ij} z^i w^j \tag{7}$$

where $\sum B_{ij} \neq 0$ then the general perturbation F_ε of F_0 can produce n small "limit cycles". Here the point $z = 0$ should not be fixed and the saddle-node bifurcation can take place.

When one considers only perturbations of (7) with zero rth jet at $z = w = 0$ then the cyclicity is $n - r - 1$.

2. Let $q = 2$. Then the "center" condition means that the map defined by the relation $F(z, w) = 0$ is the same as the map defined by the relation defined by $F(w, z) = 0$.

So, there are n cases of centers. The kth of them is defined by F's of the form $F_0 \cdot G$, where F_0 is irreducible and symmetric $F_0(z, w) = F_0(w, z)$ and $\deg F_0 = k$, $k = 1, ..., n$.

If $n = 2$,

$$F = w + \lambda z + A z^2 + B z w + C w^2$$

and $\lambda(0) = 1$ then the two center cases are each of codimension 2

$$\text{I.} \quad A - B + C = 0$$

$$\text{II.} \quad A - C = 0$$

(Here we can always assume that $z = 0$ is a fixed point of the map f). The first focus numbers are $g_0 = \lambda - 1$ and

$$g_1 = \text{const}((A - B + C)(A - C))$$

and the cyclicity of each "center" is 1.

If $n > 3$ then the calculations are more complicated.

6. APPENDIX: THE BAUTIN–DULAC THEOREM IN THE COMPLEX CASE

Probably this result was not written in the literature before. Let us formulate it.

THEOREM 7. *The maximal number of small limit cycles bifurcating from a complex saddle of the quadratic system* (2) *is* 3.

Proof. We estimate the number of fixed points of the monodromy map $\Delta_S: D_S \to D_S$, where D_S is holomorphic disc transversal to the stable separatrix $S = \{y = 0\}$ and parametrized by y (see Introduction). We have

$$\Delta_S(y) = \sum_{i=1}^{\infty} g_{i-1} y^i (1 + O(y))$$

where $g_0 = e^{-2\pi\lambda} - 1$, ($-\lambda$ is the ratio of eigenvalues for the perturbed system), and the focus quantities g_i are such polynomials of the coefficients that they vanish at the center variety. If $\lambda = 1$ then this variety is the zero set $V(\mathcal{J})$ of the ideal $\mathcal{J}(g_1, g_2, g_3)$ in $\mathbb{C}[A, B, C, D]$ generated by the focus quantities. Moreover, because of the invariance of the focus quantities with respect to the action of $\mathbb{C}^*$ (see Introduction) it is enough to consider the ideal $\mathcal{J}$ in the ring $\mathcal{R}$ of polynomials invariant with respect to changes $A_i \to e^{i\psi} A_i$, $B_i \to e^{-i\psi} B_i$, $C \to e^{-3i\psi} C$, $D \to e^{3i\psi} D$.

PROPOSITION. *The ideal $\mathcal{J}$ is radical in $\mathcal{R}$.*

Proof. In the real case this was proved in [17]. It turns out that the proof from [17] can be repeated directly in this case. ▮

Because the coefficients g_i belong to $\mathcal{R}$ and vanish at $V(\mathcal{J})$ then due to the radicality of $\mathcal{J}$ they can be represented as

$$g_i = a_{i,1} g_1 + a_{i,2} g_2 + a_{i,3} g_3$$

and we have $\Delta_S(y) - y = yF(y, \mu)$, where

$$F = g_0(1 + O(y)) + g_1 y(1 + O(y)) + g_2 y^2(1 + O(y)) + g_3 y^3(1 + O(y))$$

and $\mu = (\lambda, A_1, A_2, B_1, \ldots)$ are the parameters; μ is close to some μ_0, $\mu \in U$.

LEMMA. *The latter equation has at most* 3 *small complex zeroes.*

Proof. Bautin has proved this fact using division by non-zero functions, derivations and Rolle Lemma. Here we cannot repeat his proof and must use new arguments.

Let $\varepsilon > 0$ be a smaller number defining the radius of the disc at which we shall seek the solutions y_j of the equation $F(y, \mu) = 0$. Let $y = \varepsilon Y$.

We divide the cut neighbourhood $U \backslash \{g_0 = \cdots = g_3 = 0\}$ into several domains U_α defined by series of inequalities of the form

$$|g_i| \leqslant e^{j-i} |g_j| \qquad \text{or} \quad |g_i| > \varepsilon^{j-i} |g_j|, \qquad i < j \leqslant 3$$

In each domain U_α one of the coefficients $\varepsilon^k g_k$ in

$$F(\varepsilon Y, \mu) = g_0(1+O(\varepsilon)) + \varepsilon g_1 Y(1+O(\varepsilon)) + \varepsilon^2 g_2 Y^2(1+O(\varepsilon)) + \varepsilon^3 g_3 Y^3(1+O(\varepsilon))$$

is dominant. We divide the last function by this coefficient and obtain a deformation of the function Y^k

$$G(Y) = \sum_{k \neq i \leqslant 3} h_i Y^i(1+O(\varepsilon)) + Y^k(1+O(\varepsilon))$$

where $|h_i| < 1$. We claim that the function G has at most 3 zeroes in the disc $|Y| < 1/2$.

In order to show this claim let us consider the space $\mathbb{C}^4 = \{h(Y) = h_0 + h_1 Y + h_2 Y^2 + h_3 Y^3\}$ with the norm $\|h\| = \sup_i |h_i|$ and define on it the continuous function $H(h) = \sup_{1/2 \leqslant r \leqslant 1} \inf_\theta |h(re^{i\theta})|$. There is a constant $C > 0$ such that $H|_{\|h\|=1} > C$. This means that for any polynomial $h(Y)$ with $\sup |h_i| = 1$ there is a circle $|Y| = r$, $1/2 \leqslant r = r(h) \leqslant 1$ on which $|h(Y)| > C$.

If ε sufficiently small then the function $G(Y) = h(Y) + O(\varepsilon)$ is nonzero along $|Y| = r(h)$. Thus we can apply the argument principle which says that the number of zeroes of G inside the disc $|Y| < r(h)$ is the same as the number of zeroes of $h(Y)$ in this disc. ∎

Thus we have the upper bound for the number of limit cycles. The lower bound was proved by Bautin.

ACKNOWLEDGMENTS

Part of these results were obtained during the author's visit at Université de Paris VI. He thanks the University for its hospitality and support. Discussions with J.-P. Françoise and A. Fronville were very fruitful.

REFERENCES

1. V. I. Arnold and Yu S. Il'yashenko, Ordinary differential equations, *Fund. Direction* **1** (1985), 1–146.
2. G. Bateman and A. Erdély, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.
3. N. N. Bautin, On the number of limit cycles which appear with the variation of the coefficients from an equilibrium point of focus or center type, *Math. Sbornik*. **30** (1952), 181–196 [in Russian].
4. F. S. Berezovskaya and N. B. Medvedeva, The asymptotic of the return map of a singular point with fixed Newton diagram, *Trans. Petrovski Sem.* **15** (1991), 156–177 [in Russian].

5. D. Cerveau and R. Moussu, Groupes d'automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $y\,dy + \cdots$, *Bull. Soc. Math. France* **116** (1988), 459–488.
6. H. Dulac, Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre, *Bull. Sci. Math.* **32** (2) (1908), 230–252.
7. P. M. Elizarov, Yu S. Il'yashenko, A. A. Scherbakov, and S. M. Voronin, Finitely generated groups of germs of one-dimensional conformal mappings and invariants for complex singular point of analytic foliations of the complex plane, Nonlinear Stokes Phenomena, *Adv. Soviet Math.* **14** (1993), 57–105.
8. A. Fronville, Algorithmic approach of the center problem for $1 : -2$ resonant singular points of polynomial vector fields, to appear.
9. Yu S. Il'yashenko, Algebraic non-solvability and almost algebraic solvability of the center-focus problem, *Funct. Anal. Appl.* **6** (1972), 30–37 [in Russian].
10. W. Kapteyn, Nieuw ordeozec omtrent de middlepunten der integralen van differentiaalvergelijkingen van de eerste orde en den eersten graad, *Koninkl. Nederland Akad.* **20** (1912), 1354–1365; **21** (1912), 27–33.
11. A. M. Lyapunov, "General Stability of Motion," Gostekhizdat, Moscow, 1950 [in Russian].
12. H. Poincaré, "Mémoire sur les courbes définies par uneéquation différentielle," Oeuvre de H. Poincaré, Vol. 1, Gauthiers-Villars, Paris, 1951.
13. V. G. Romanovski, e-mail correspondence.
14. A. Seidenberg, Reduction of differential equation $A\,dx = B\,dy$, *Amer. J. Math.* **90** (1968), 248–269.
15. J. Sokulski, The beginning of classification of Darboux integrals for cubic systems with center, *J. Integr. Differential Equations*, submitted.
16. J. Sokulski, Algebraiczne krzywe niezmiennicze i warunki na centrum dla wielomianowych pól wektorowych, Ph.D. thesis, Polish Acad. of Sciences, 1996.
17. H. Żołądek, Quadratic systems with center and their perturbations, *J. Differential Equations* **109** (1994), 223–273.
18. H. Żołądek, The classification of reversible cubic systems with center, *Topol. Methods Nonlinear Anal.* **4** (1994), 79–136.
19. H. Żołądek, Eleven small limit cycles in a cubic vector field, *Nonlinearity* **8** (1995), 843–860.